Lemma 11 and Curvature

> Moreover, in all of these we have supposed the angle of contact to be neither infinitely greater than the angles of contact which circles contain with their tangents, nor infinitely less than the same; that is, the curvature at point A to be neither infinitely small nor infinitely large, or the interval AJ to be of a finite magnitude. (Newton, 31)

Immediately after his Lemmas, Newton makes some striking claims about curvature:

1. These Lemmas only work for finite curves
2. Infinite curves have an "angle of contact" either infinitely greater or infinitely less than that of a circle
3. "The interval AJ" will not have a finite magnitude.

These are striking and tremendously important claims for which Newton offers no argument, but they can be easily demonstrated using a general understanding of Cartesian curves using concept of a limit.

First, we must show how to find AJ for a given function:

Taking the diagram from Lemma 11, let the tangent and GB both be extended until they meet at C, and draw BD perpendicular to the tangent.

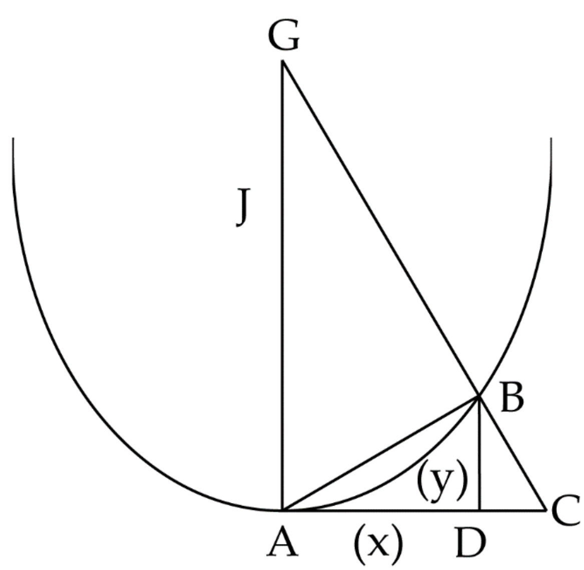


Figure 1

Then ADB, BDC and GAC will all be similar triangles.

Let AD be called "x" and BD be called "y", just as they would be if this curve was plotted on a Cartesian coordinate graph if AD was a line on the X axis with A at the origin.

∴ DC : y :: y : x,

And GA : x + DC :: x : y.

Expressing these as the equivalent equations, $\frac{DC}{y} = \frac{y}{X}$

and $\frac{GA}{x+DC} = \frac{x}{y}$.

$\therefore DC = \frac{y^2}{x}$

and $GA = \frac{x^2}{y} + \frac{DCx}{y}$

$\therefore GA = \frac{x^2}{y} + \frac{(\frac{y^2}{x})(x)}{y} = \frac{x^2}{y} + \frac{xy^2}{xy} = \frac{x^2}{y} + y.$

But $AJ = \lim_{x \to 0} AG$

$\therefore AJ = \lim_{x \to 0} \frac{x^2}{y} + y = \lim_{x \to 0} \frac{x^2}{y} + \lim_{x \to 0} y$

But x is tangent to the curve at the origin, so as it goes to zero, so does y.

$\therefore AJ = \lim_{x \to 0} \frac{x^2}{y}.$

QEI

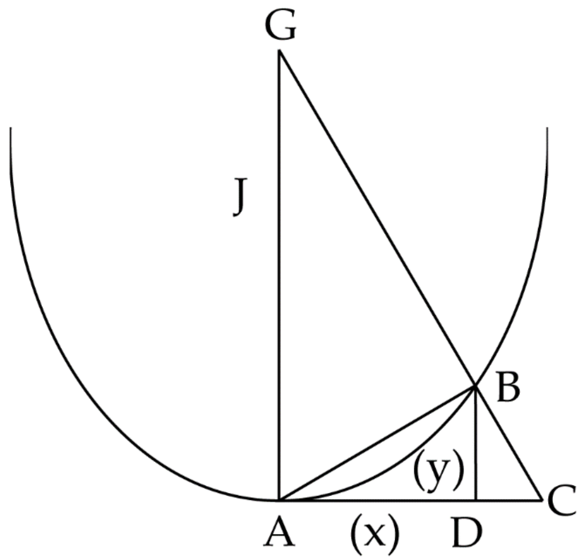


Figure 2

From this it is clear that for any function of x to have a finite AJ at the point where it is tangent to the x axis, the whole function must have a finite limit of $\frac{x^2}{y}$ as x goes to zero. For instance, the function $y = x^2$ might be given. In this case, $AJ = \lim_{x \to 0} \frac{x^2}{x^2} = 1$. Since the x^2 terms cancel out, the function is the same everywhere and we are given a finite limit. If Descartes' canonical equations for the several conic sections were used, it could be proven using this method that they all will have a finite AJ at their vertices. This proof applies directly when the x axis is tangent to the conic section, and since any point can be one of their vertices, it can also be proven that every conic section has a finite AJ at every point. If, on the other hand, we were given that $y = x^4$, then $AJ = \lim_{x \to 0} \frac{x^2}{x^4} = \lim_{x \to 0} x^{-2} = \infty$; or, if $y = x^{\frac{2}{3}}$, then $AJ = \lim_{x \to 0} \frac{x^2}{x^{\frac{2}{3}}} = \lim_{x \to 0} x^{1\frac{1}{3}} = 0$. This gives us an example of a function with an infinite AJ ($y = x^4$), a function with a finite AJ ($y = x^2$), and a function with a nil AJ ($y = x$). Now let's put this together into some general rules:

When dividing exponents with the same root, the exponent in the denominator is subtracted from the exponent in the numerator. When calculating AJ, our numerator will always be x2, and the denominator will always be the power of x that is equal to y (multiplied by a variable if there are any). From this it follows that any positive power of x goes to zero as x goes to zero, and any negative power of x goes to infinity as x goes to zero. Thus, if we are given a constant multiplied by any power of x, AJ will always be inversely as that constant if that power of x is 2; it will be infinite if the power of x is greater than 2; and AJ will be nil if the power of x is less than 2.

In other words: if $y = kx^{n>2}$, then $AJ = \infty$

If $y = kx^{n<2}$, then $AJ = 0$.

If $y = kx^2$, then $AJ = \frac{1}{k}$.

This gives us an answer for why it is that AJ does not have a finite value in any of the curves that Newton mentioned, and why every conic section does have a finite AJ at every point.

However, to proceed with our discussion of curvature, we must first set forth two postulates.

Two things must be postulated to do this:

1. Any curve with a greater AJ is less curved at A; any curve with a lesser AJ is more curved at A; any curves with equal AJ's have equal curvature at A.
2. Curvature is the rate of change of the angle made by the tangent-line with any fixed line over a certain circumferential distance.

For instance, since every circle has a tangent line that makes one full rotation at a constant rate over the course of the whole circumference, the angle between that tangent and any fixed line will change by 360° over the course of the whole rotation, the curvature of any point on any circle will always be inversely proportional to its circumference.

And since every semi-circle is similar, the curvature of any circle is also inversely proportional to its diameter.

The diameter of every circle is always equal to its AJ, since the axis perpendicular to the tangent is always the diameter and the angle in a semicircle is always right.

∴ The curvature of every circle is inversely proportional to its AJ, which is its diameter.

However, whenever two curves have an equal AJ, they have equal curvature, so whenever any curve has an AJ equal to the diameter of a circle, its curve is the same as the curve of that circle.

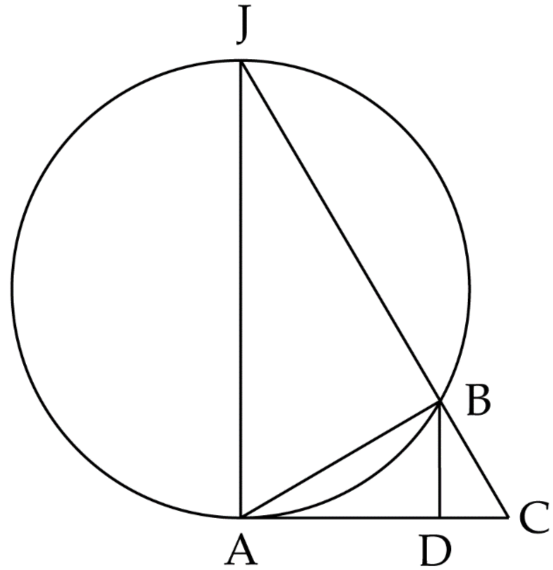
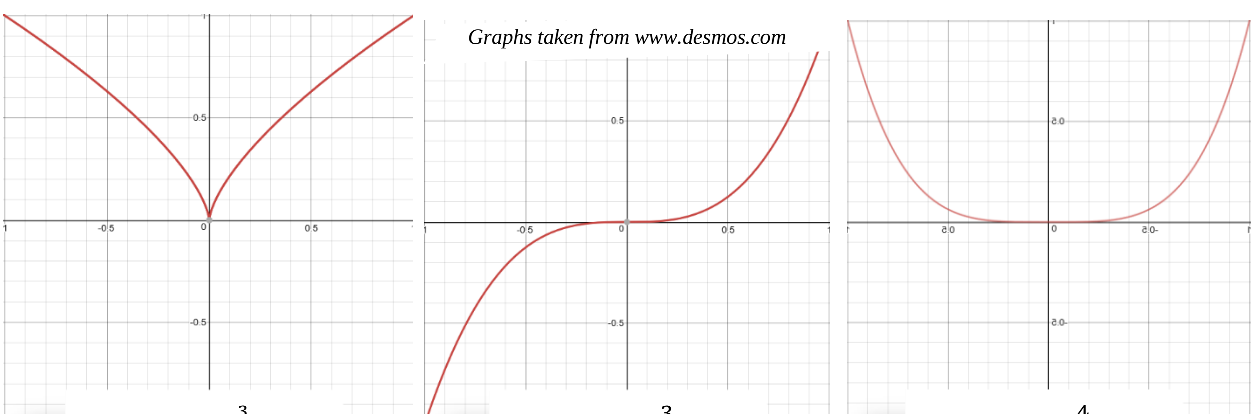


Figure 3

Since the curvature of any circle is inversely proportional to its AJ, ***it follows that the curvature at any vertex "A" is always inversely proportional to its AJ.***

It has already been shown that every function of x that is raised to a greater power than 2 will have an infinite AJ and that every function of x to a lesser power of 2 has a nil AJ. Therefore, at the vertex of a function whose power is greater than 2, there will be infinitely little curvature, and at the vertex of any function whose power is less than 2 there will be infinitely great curvature.



Graphs taken from www.desmos.com

$y = x^{\frac{3}{2}}$ $\qquad$ $y = x^3$ $\qquad$ $y = x^4$

From left to right these show the graph of y to the sesquiplicate ratio (infinite curvature), the triplicate ratio (no curvature) and the quadruplicate ratio (no curvature).

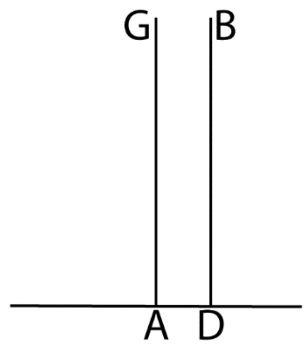


Figure 4

Keeping these graphs in mind, let us now turn to a more theoretical consideration of what would be required for the limit of AG to be finite.

Given AD as a straight line (Figure 4), AG is always perpendicular to AD and DB is always parallel to AG. Since these lines will not cut each other before coinciding with each other, their ultimate length (AJ) will be infinite. Since the curvature is inversely proportional to AJ, the curvature is $1/\infty$.

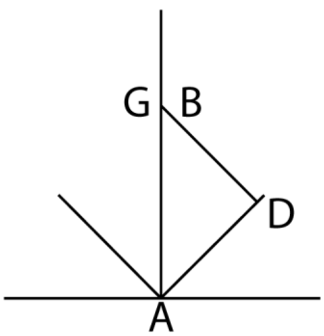


Figure 5

If on the other hand we are given two straight lines making an angle at the origin (Figure 5), AG is cut away in a constant proportion with AD. When AD goes to zero AG will go to zero. This causes AJ to be nil. This is because there is an infinite number of tangents to an angle This is what makes it have infinitely great curvature, as the tangent rotates a finite degree at a point. For any circle to have curvature equal to an angle, that circle would need to have a diameter of infinitely small length.

This shows the reason that the three graphs earlier used as examples do not have finite curvature, but why say that they are curved at all? The answer is simple: straight lines are drawn between two points, but pick any two points on the x^4 graph, join a line between them, and that line will not coincide with the graph. Similarly put, there is an inclination between the x^4 curve and the x axis, since they coincide only at the origin. Since there is an inclination there is an angle, but this angle is clearly not a rectelineal angle, for then a rectelineal angle would be smaller than a horn angle. This too must be a horn angle then, but where there is a horn angle there must be a curve.

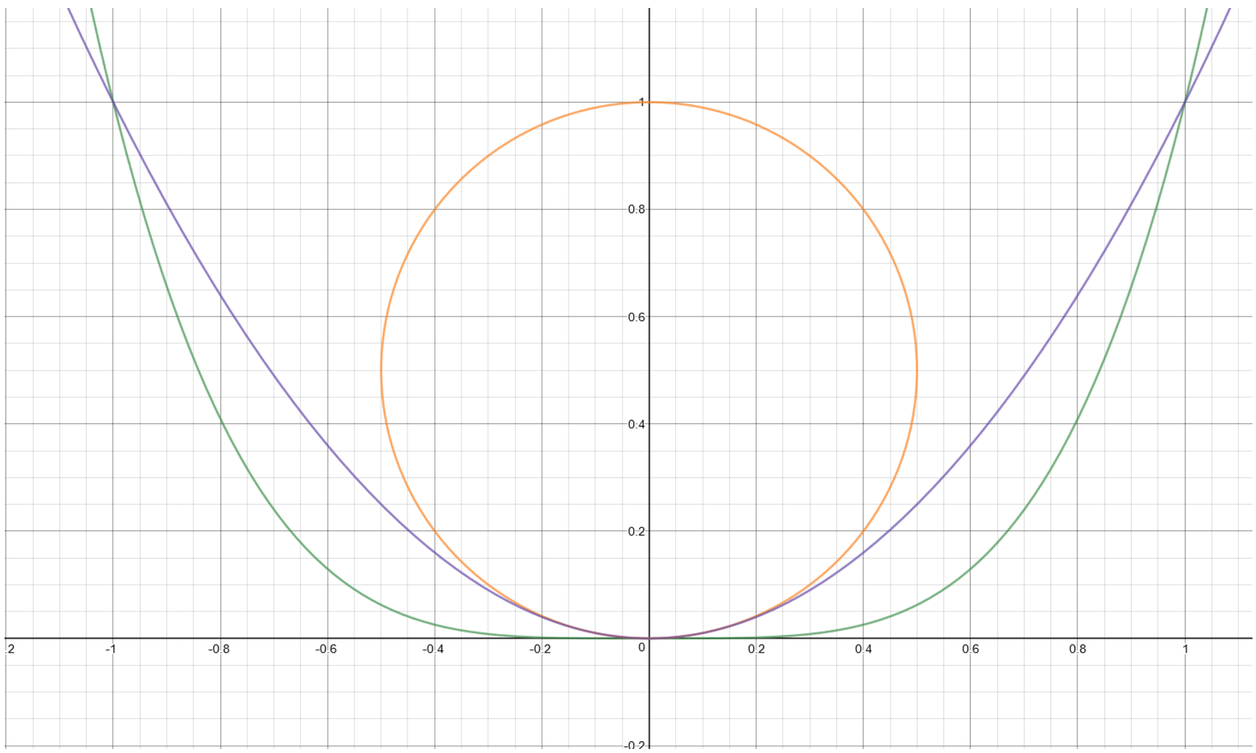


Figure 6